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## Topological classification of chains of linear mappings

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## ABSTRACT

We consider systems of linear mappings  $A_1, \dots, A_{t-1}$  of the form

$$\mathcal{A} : U_1 \xrightarrow{A_1} U_2 \xrightarrow{A_2} U_3 \xrightarrow{A_3} \dots \xrightarrow{A_{t-1}} U_t$$

in which  $U_1, \dots, U_t$  are unitary (or Euclidean) spaces and each line is either the arrow  $\longrightarrow$  or the arrow  $\longleftarrow$ . Let  $\mathcal{A}$  be transformed to

$$\mathcal{B} : V_1 \xrightarrow{B_1} V_2 \xrightarrow{B_2} V_3 \xrightarrow{B_3} \dots \xrightarrow{B_{t-1}} V_t$$

by a system  $\{\varphi_i : U_i \rightarrow V_i\}_{i=1}^t$  of bijections. We say that  $\mathcal{A}$  and  $\mathcal{B}$  are linearly isomorphic if all  $\varphi_i$  are linear. Considering all  $U_i$  and  $V_i$  as metric spaces, we say that  $\mathcal{A}$  and  $\mathcal{B}$  are topologically isomorphic if all  $\varphi_i$  and  $\varphi_i^{-1}$  are continuous.

We prove that  $\mathcal{A}$  and  $\mathcal{B}$  are topologically isomorphic if and only if they are linearly isomorphic.

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## 1. Introduction and theorem

A chain of linear mappings is a system of linear mappings  $A_1, \dots, A_{t-1}$  of the form

$$\mathcal{A} : U_1 \xrightarrow{A_1} U_2 \xrightarrow{A_2} U_3 \xrightarrow{A_3} \dots \xrightarrow{A_{t-1}} U_t \quad (1)$$

in which each line is either the arrow  $\longrightarrow$  or the arrow  $\longleftarrow$ . We assume that  $U_1, \dots, U_t$  are unitary spaces (or are Euclidean spaces). Without loss of generality, the reader may think that all  $U_1, \dots, U_t$  are  $\mathbb{C} \oplus \dots \oplus \mathbb{C}$  (or  $\mathbb{R} \oplus \dots \oplus \mathbb{R}$ , respectively) with a natural topology on them.

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Let

$$\mathcal{B} : V_1 \xrightarrow{B_1} V_2 \xrightarrow{B_2} V_3 \xrightarrow{B_3} \dots \xrightarrow{B_{t-1}} V_t \quad (2)$$

be a chain with the same orientation of arrows as in (1). We write  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$  if  $\varphi = \{\varphi_i : U_i \rightarrow V_i\}_{i=1}^t$  is a system of bijections such that all squares in the diagram

$$\begin{array}{ccccccccccc} U_1 & \xrightarrow{A_1} & U_2 & \xrightarrow{A_2} & U_3 & \xrightarrow{A_3} & \dots & \xrightarrow{A_{t-2}} & U_{t-1} & \xrightarrow{A_{t-1}} & U_t \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & & & \varphi_{t-1} \downarrow & & \varphi_t \downarrow \\ V_1 & \xrightarrow{B_1} & V_2 & \xrightarrow{B_2} & V_3 & \xrightarrow{B_3} & \dots & \xrightarrow{B_{t-2}} & V_{t-1} & \xrightarrow{B_{t-1}} & V_t \end{array}$$

are commutative; that is,

$$\begin{aligned} \varphi_{i+1}A_i &= B_i\varphi_i & \text{if } A_i : U_i &\rightarrow U_{i+1}, \\ \varphi_iA_i &= B_i\varphi_{i+1} & \text{if } A_i : U_i &\leftarrow U_{i+1} \end{aligned}$$

for each  $i = 1, \dots, t-1$ .

**Definition.** We say that  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$  is

- (i) an *isometry* if each  $\varphi_i : U_i \rightarrow V_i$  is a linear bijection that preserves the scalar product; that is, each  $\varphi_i$  is a unitary map (or an orthogonal map if all spaces are Euclidean);
- (ii) a *linear isomorphism* if each  $\varphi_i : U_i \rightarrow V_i$  is a linear bijection (in this definition, we forget that  $U_i$  and  $V_i$  are metric spaces and consider them as linear spaces);
- (iii) a *topological isomorphism* if each  $\varphi_i : U_i \rightarrow V_i$  is a homeomorphism, which means that  $\varphi_i$  and  $\varphi_i^{-1}$  are continuous and bijective (we forget that  $U_i$  and  $V_i$  are linear spaces and consider them as metric spaces).

Each linear bijection of unitary (or Euclidean) spaces is a homeomorphism, hence

$$\begin{aligned} \varphi : \mathcal{A} &\xrightarrow{\sim} \mathcal{B} \text{ is an isometry} \\ &\implies \varphi \text{ is a linear isomorphism} \\ &\implies \varphi \text{ is a topological isomorphism.} \end{aligned}$$

The main result of this paper is the following theorem, which is proved in Section 4.

**Theorem 1.** Two chains of linear mappings on unitary (or Euclidean) spaces are topologically isomorphic if and only if they are linearly isomorphic.

Note that the problem of topological classification was also studied for linear operators [6–8, 11, 13, 4] (Budnitska is the maiden name of the first author), affine operators [2, 9, 3–5], dynamical systems [14], and representations of Lie groups [15].

The paper is organized as follows. In Section 2 we show that the problem of classifying chains (1) up to isometry is hopeless for each  $t \geq 3$ . In Section 3 we recall a known classification of chains (1) up to linear isomorphism; we formulate it in terms of dimensions of some subspaces. In Section 4 we show that these dimensions are also topological invariants, which proves Theorem 1.

## 2. Isometry of chains

In this section, we consider chains (1) of linear mappings on *unitary* spaces. It would be the most natural to classify them up to isometry.

If  $t = 2$ , then the classification of chains (1) up to isometry is given by the singular value decomposition: there exist orthonormal bases in  $U_1$  and  $U_2$  in which the matrix of  $A_1$  is  $\text{diag}(a_1, \dots, a_r) \oplus 0$ , where  $a_1 \geq \dots \geq a_r > 0$  are real numbers that are uniquely determined by  $A_1$ .

Unfortunately, the problem of classifying chains up to isometry must be considered as hopeless for  $t = 3$  (and so for each  $t \geq 3$ ) since it contains the problem of classifying linear operators on unitary spaces up to unitary similarity, and hence all systems of linear mappings on unitary spaces (see the end of this section). This statement is proved sketchily in [16, Section 2.3]; for the reader convenience we prove in detail the following weaker assertion.

**Theorem 2.** *The problem of classifying chains*

$$U_1 \xrightarrow{A_1} U_2 \xleftarrow{A_2} U_3, \quad U_1, U_2, U_3 \text{ are unitary spaces}, \quad (3)$$

*up to isometry contains the problem of classifying linear operators on unitary spaces up to unitary similarity.*

**Proof.** We say that matrices  $X$  and  $Y$  are *unitarily similar* if there exists a unitary matrix  $S$  such that  $S^{-1}XS = Y$ .

Let us consider chains of mappings

$$U_1 \xrightarrow{A_1} U_2 \xleftarrow{A_2} U_3 \quad \text{and} \quad V_1 \xrightarrow{B_1} V_2 \xleftarrow{B_2} V_3 \quad (4)$$

that are given in some orthonormal bases by pairs of matrices  $(M, N_X)$  and  $(M, N_Y)$  in which

$$M := \begin{bmatrix} I & 0 & 0 \\ 0 & 2I & 0 \\ 0 & 0 & 3I \end{bmatrix}, \quad N_X := \begin{bmatrix} I & 0 \\ I & I \\ I & X \end{bmatrix}, \quad N_Y := \begin{bmatrix} I & 0 \\ I & I \\ I & Y \end{bmatrix},$$

and all blocks are  $m \times m$ .

It suffices to prove that

$$\text{the chains (4) are isometric if and only if } X \text{ and } Y \text{ are unitarily similar.} \quad (5)$$

Indeed, assume we know a set of canonical matrix pairs for (3). We take those of them that can be reduced to the form  $(M, N_X)$  and reduce them to it. Due to (5), the obtained blocks  $X$  form a set of canonical matrices for unitary similarity.

Let us prove (5).

“ $\implies$ ” Let the chains (4) be isometric; that is, there exist unitary matrices  $S_1, S_2, S_3$  such that

$$S_2^{-1}MS_1 = M, \quad S_2^{-1}N_XS_3 = N_Y. \quad (6)$$

By the first equality in (6),

$$S_1^*M^*S_2 = M^*, \quad MM^*S_2 = MS_1M^* = S_2MM^*.$$

Since  $MM^* = I_m \oplus 4I_m \oplus 9I_m$ , we have  $S_2 = C_1 \oplus C_2 \oplus C_3$  for some  $m \times m$  matrices  $C_1, C_2$ , and  $C_3$ .

By the second equality in (6),  $S_2N_Y = N_XS_3$ . Equating the corresponding horizontal strips, we obtain

$$\begin{bmatrix} C_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} S_3, \quad \begin{bmatrix} C_2 & C_2 \end{bmatrix} = \begin{bmatrix} I & I \end{bmatrix} S_3, \quad \begin{bmatrix} C_3 & C_3Y \end{bmatrix} = \begin{bmatrix} I & X \end{bmatrix} S_3. \quad (7)$$

Let  $S_3 = [R_{ij}]_{i,j=1}^2$ . The first equality in (7) implies that  $R_{11} = C_1$  and  $R_{12} = 0$ . Since  $S_3$  is unitary,  $R_{21} = 0$  and so  $S_3 = C_1 \oplus R_{22}$ . The second equality in (7) implies that  $C_1 = C_2 = R_{22}$ . The third equality in (7) implies that  $C_3 = C_2 = R_{22}$  and  $C_3Y = XC_3$ . Thus,  $X$  and  $Y$  are unitarily similar.

“ $\impliedby$ ” Conversely, if  $C^{-1}XC = Y$  for some unitary  $C$ , then (6) holds for  $S_1 = S_2 = C \oplus C \oplus C$  and  $S_3 = C \oplus C$ , and so the chains (4) are isometric.  $\square$

Recall that a *quiver* is a directed graph. Its *representation* is given by assigning to each vertex a unitary space and to each arrow a linear mapping of the corresponding vector spaces. A representation is *unitary* if all of its vector spaces are unitary.

It was shown in [16, Section 2.3] that the problem of classifying linear operators on unitary spaces up to unitary similarity contains the problem of classifying unitary representations of an *arbitrary* quiver. Thus, we cannot expect to find an observable system of invariants for linear operators on unitary spaces. Nevertheless, we can reduce the matrix of *any given* linear operator on a unitary space (moreover, the matrices of any given unitary representation of a quiver) to canonical form by using Littlewood's algorithm; see [16, Section 3].

In the same way, the problem of classifying pairs of linear operators on a vector space is considered as hopeless (and all classification problems that contain it are called *wild*) since it contains the problem of classifying representations of each quiver. Nevertheless, we can reduce the matrices of any given representation of a quiver to canonical form by using Belitskii's algorithm; see [1, 17].

### 3. Linear isomorphism of chains

In this section, we consider chains of linear mappings

$$\mathcal{A} : U_1 \xrightarrow{A_1} U_2 \xrightarrow{A_2} U_3 \xrightarrow{A_3} \dots \xrightarrow{A_{t-1}} U_t \quad (8)$$

on vector spaces without scalar product. Without complicating the proofs, we consider them over *any* field  $\mathbb{F}$ . In Theorem 3 we recall the well-known classification of such chains up to linear isomorphisms (see Definition 1(ii)). Next we fix some subspaces of  $U_1, \dots, U_t$  and prove in Theorem 4 that the set of their dimensions is a full system of invariants of chains with respect to linear isomorphisms. In Section 4 we establish that this set is also a full system of invariants of chains with respect to topological isomorphisms, which proves Theorem 1.

#### 3.1. A classification of chains up to linear isomorphisms

The directions ( $U_i \rightarrow U_{i+1}$  or  $U_i \leftarrow U_{i+1}$ ) of all linear mappings  $A_i$  in (8) can be given by the directed graph

$$G : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{t-1}} t \quad (9)$$

in which each arrow  $\alpha_i$  is directed as  $A_i$ . (Thus, each chain (8) defines a representation of the quiver (9) and a linear isomorphism of chains defines an isomorphism of the corresponding representations.) Write

$$\mathcal{A}(k) := U_k, \quad k = 1, \dots, t. \quad (10)$$

The *direct sum* of chains  $\mathcal{A}$  and

$$\mathcal{B} : V_1 \xrightarrow{B_1} V_2 \xrightarrow{B_2} V_3 \xrightarrow{B_3} \dots \xrightarrow{B_{t-1}} V_t$$

with the same directed graph (9) is the chain

$$\mathcal{A} \oplus \mathcal{B} : U_1 \oplus V_1 \xrightarrow{A_1 \oplus B_1} U_2 \oplus V_2 \xrightarrow{A_2 \oplus B_2} \dots \xrightarrow{A_{t-1} \oplus B_{t-1}} U_t \oplus V_t.$$

For every pair of integers  $(i, j)$  such that  $1 \leq i \leq j \leq t$ , we define the chain

$$\mathcal{L}_{ij} : 0 - \dots - 0 - \mathbb{F} \xrightarrow{1} \mathbb{F} \xrightarrow{1} \dots \xrightarrow{1} \mathbb{F} - 0 - \dots - 0$$

in which “1” is the identity bijection and  $\mathbb{F}$ 's are at the vertices  $i, i+1, \dots, j$  of (9).

The following theorem is well known in the theory of quiver representations; the representations of (9) and the other quivers that have a finite number of nonisomorphic indecomposable representations were classified by Gabriel [10].

**Theorem 3.** *Each chain  $\mathcal{A}$  is linearly isomorphic to a direct sum of chains of the form  $\mathcal{L}_{ij}$ . This direct sum is uniquely determined by  $\mathcal{A}$ , up to permutation of summands.*

An algorithm for constructing this canonical form of chains of linear mappings over  $\mathbb{C}$  is given in [18, Section 4]; it uses only transformations of unitary equivalence of matrices:  $M \mapsto S_1 M S_2$  in which  $S_1$  and  $S_2$  are unitary.

**Corollary of Theorems 1 and 3.** *Each chain  $\mathcal{A}$  of linear mappings on unitary (or Euclidean) spaces is topologically isomorphic to a direct sum of chains of the form  $\mathcal{L}_{ij}$ . This direct sum is uniquely determined by  $\mathcal{A}$ , up to permutation of summands.*

Let  $\mathcal{A}$  be any chain of the form (8). In each of its spaces  $U_i$ , we define a series of subspaces

$$0 = U_{i0} \subset U_{i1} \subset U_{i2} \subset \cdots \subset U_{it} = U_i, \quad i = 1, \dots, t \quad (11)$$

by induction:  $0 = U_{i0} \subset U_{i1} = U_i$  and if (11) is constructed for  $i < t$  then

$$(U_{i+1,1}, \dots, U_{i+1,i}) := \begin{cases} (A_i U_{i1}, \dots, A_i U_{it}) & \text{if } A_i : U_i \rightarrow U_{i+1}, \\ (\text{Ker } A_i, A_i^{-1} U_{i1}, \dots, A_i^{-1} U_{i,i-1}) & \text{if } A_i : U_i \leftarrow U_{i+1} \end{cases} \quad (12)$$

(here  $A_i^{-1} U_{ij}$  denotes the preimage of  $U_{ij}$ ).

### 3.2. An example

Each chain of the form

$$\mathcal{A} : U_1 \xrightarrow{A_1} U_2 \xleftarrow{A_2} U_3 \quad (13)$$

is given by the pair of matrices  $(M_1, M_2)$  in some bases of  $U_1, U_2, U_3$ . Changing the bases, we can reduce the pair by transformations

$$(M_1, M_2) \mapsto (S_2^{-1} M_1 S_1, S_2^{-1} M_2 S_3), \quad S_1, S_2, S_3 \text{ are nonsingular.} \quad (14)$$

It is convenient to give  $(M_1, M_2)$  by the block matrix  $[M_1 | M_2]$  since the rows of  $M_1$  and  $M_2$  are transformed by the same matrix  $S_2^{-1}$ . Due to (14), we can reduce it by elementary row transformations (i.e., by simultaneous elementary transformations with rows of  $M_1$  and  $M_2$ ) and by elementary column transformations within  $M_1$  and  $M_2$ . Each  $[M_1 | M_2]$  can be reduced by these transformations to its canonical form

$$\boxed{N_1 | N_2} = \begin{array}{|cc|cc|cc|} \hline & & 0 & I_p & 0 & I_r & 0 \\ & & & & 0 & 0 & 0 \\ \hline & & 0 & 0 & 0 & & I_q \\ & & & & 0 & & 0 \\ \hline \end{array} \quad (15)$$

as follows (see [18, Section 4]). We first reduce  $M_1$  to the form

$$\begin{array}{cc} 0 & I \\ 0 & 0 \end{array} \quad (16)$$

and denote the obtained block matrix by  $[N_1 | M'_2]$ . Then we extend to  $M'_2$  the partition of  $N_1$  into two horizontal strips and reduce the second horizontal strip of  $M'_2$  to the form (16):

$$\begin{array}{cc|cc} 0 & I_p & M_{11} & M_{12} \\ \hline 0 & 0 & 0 & I_q \\ & & 0 & 0 \end{array}$$

We make  $M_{12}$  equal to zero by adding linear combinations of rows of  $I_q$ . At last, we reduce  $M_{11}$  to the form (16) by elementary transformations; these transformations may spoil  $I_p$ , we restore it by column transformations. The obtained block matrix has the form (15).

For example, let the chain (13) be given in some bases  $\{e_i\}_{i=1}^5$ ,  $\{f_i\}_{i=1}^6$ , and  $\{g_i\}_{i=1}^5$  of  $U_1$ ,  $U_2$ , and  $U_3$  by the following canonical block matrix of the form (15):

$$\begin{array}{c|cccccc|ccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & g_1 & g_2 & g_3 & g_4 & g_5 \\ \hline f_1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ f_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ f_3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ f_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ f_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ f_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Then  $\mathcal{A}$  is the direct sum of 9 chains that are given by the action of the mappings on the basic vectors as follows:

$$\begin{array}{l} e_1 \longrightarrow 0 \longleftarrow 0 \\ e_2 \longrightarrow 0 \longleftarrow 0 \\ 0 \longrightarrow 0 \longleftarrow g_1 \\ e_3 \longrightarrow f_1 \longleftarrow g_2 \\ e_4 \longrightarrow f_2 \longleftarrow g_3 \\ e_5 \longrightarrow f_3 \longleftarrow 0 \\ 0 \longrightarrow f_4 \longleftarrow g_4 \\ 0 \longrightarrow f_5 \longleftarrow g_5 \\ 0 \longrightarrow f_6 \longleftarrow 0 \end{array}$$

(For simplicity of notation, we write  $0 \mapsto f_i$  instead of  $0 \mapsto 0$ .) Thus,  $\mathcal{A}$  is linearly isomorphic to

$$\mathcal{L}_{11} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{33} \oplus \mathcal{L}_{13} \oplus \mathcal{L}_{13} \oplus \mathcal{L}_{12} \oplus \mathcal{L}_{23} \oplus \mathcal{L}_{23} \oplus \mathcal{L}_{22}.$$

The subspaces  $U_{ij}$  defined in (11) and (12) are the following:

$$\begin{array}{l} U_{10} = 0, \quad U_{11} = U_1; \\ U_{20} = 0, \quad U_{21} = \langle f_1, f_2, f_3 \rangle, \quad U_{22} = U_2; \\ U_{30} = 0, \quad U_{31} = \langle g_1 \rangle, \quad U_{32} = \langle g_1, g_2, g_3 \rangle, \quad U_{33} = U_3; \end{array}$$

here  $\langle x, y, \dots, z \rangle$  denotes the subspace spanned by  $x, y, \dots, z$ .

Note that

$$U_{32} = U_{31} \oplus \langle g_2 \rangle \oplus \langle g_3 \rangle \quad (17)$$

in which  $\langle g_2 \rangle$  and  $\langle g_3 \rangle$  are the vector spaces of the chains given by

$$e_3 \mapsto f_1 \leftarrow g_2 \quad \text{and} \quad e_4 \mapsto f_2 \leftarrow g_3$$

### 3.3. A system of invariants

**Theorem 4.** Each chain  $\mathcal{A}$  is fully determined, up to linear isomorphism, by the indexed set

$$\{n_{ij}\}_{1 \leq j \leq i \leq t} \quad \text{in which } n_{ij} := \dim U_{ij} \quad (18)$$

and  $U_{ij}$  are defined in (12).

**Proof.** By Theorem 3,  $\mathcal{A}$  possesses a canonical decomposition

$$\mathcal{A} := \bigoplus_{\ell=1}^s \mathcal{A}_\ell, \quad \mathcal{A}_\ell \simeq \mathcal{L}_{p_\ell q_\ell}, \quad (19)$$

whose summands are determined up to renumbering and linear isomorphisms of summands. Thus,  $\mathcal{A}$  is determined up to linear isomorphism by the family of pairs  $\{(p_\ell, q_\ell)\}_{\ell=1}^s$  and this family is determined by  $\mathcal{A}$  up to renumbering (i.e.,  $\{(p_\ell, q_\ell)\}_{\ell=1}^s$  is an unordered set with repeating elements).

For technical reason, it is better to prove the following statements that are stronger than the theorem:

- (i)  $\{n_{ij}\}_{1 \leq j \leq i \leq t}$  uniquely determines  $\{(p_\ell, q_\ell)\}_{\ell=1}^s$ , up to renumbering,
- (ii) there are indices  $\ell(i, j) \in \{1, \dots, s\}$  such that each of the spaces  $U_{t1}, \dots, U_{tt}$  defined in (11) is decomposed into the direct sum

$$U_{ti} = U_{t,i-1} \oplus \mathcal{A}_{\ell(i,1)}(t) \oplus \dots \oplus \mathcal{A}_{\ell(i,r_i)}(t) \quad (20)$$

(see (10); we put  $r_i := 0$  if  $U_{t,i-1} = U_{ti}$ ).<sup>1</sup>

- (iii) all chains  $\mathcal{A}_{\ell(i,1)}, \dots, \mathcal{A}_{\ell(i,r_i)}$  have the first nonzero space at the same position, i.e.

$$p_{\ell(i,1)} = \dots = p_{\ell(i,r_i)} =: a_i,$$

- (iv)  $a_i \neq a_j$  if  $i \neq j$ .

We use induction on  $t$ . The induction base is trivial: the statements (i)–(iv) hold for chains with 2 vector spaces; that is, for  $U_1 \xrightarrow{A_1} U_2$  and  $U_1 \xleftarrow{A_1} U_2$ .

Suppose that (i)–(iv) hold for chains with  $t - 1$  vector spaces, in particular, for the restriction

$$\mathcal{A}' : U_1 \xrightarrow{A_1} U_2 \xrightarrow{A_2} U_3 \xrightarrow{A_3} \dots \xrightarrow{A_{t-2}} U_{t-1}$$

of  $\mathcal{A}$  to the first  $t - 1$  spaces. We can suppose that the summands in (19) are numbered such that

$$\max(p_1, \dots, p_{s'}) < t = p_{s'+1} = \dots = p_s. \quad (21)$$

The canonical decomposition of  $\mathcal{A}'$  can be obtained from (19) as follows:

$$\mathcal{A}' := \bigoplus_{\nu=1}^{s'} \mathcal{A}'_\nu, \quad \mathcal{A}'_\nu \simeq \mathcal{L}_{p_\nu q'_\nu}, \quad q'_\nu := \min(t - 1, q_\nu),$$

in which  $s'$  is defined in (21) and every  $\mathcal{A}'_\nu$  is the restriction of  $\mathcal{A}_\nu$  to the first  $t - 1$  vector spaces.

By induction hypothesis,

<sup>1</sup> An example of this decomposition is given in (17), in which  $t = 3$ ,  $i = r_i = 2$ ,  $\mathcal{A}_{\ell(2,1)}(3) = \langle g_2 \rangle$ , and  $\mathcal{A}_{\ell(2,2)}(3) = \langle g_3 \rangle$ .

- $\{n_{ij}\}_{1 \leq j \leq i \leq t-1}$  uniquely determines  $\{(p_v, q'_v)\}_{v=1}^{s'}$ , up to renumbering,
- there are indices  $v(i, j) \in \{1, \dots, s'\}$  such that each of the spaces  $U_{t-1,1}, \dots, U_{t-1,t-1}$  is decomposed into the direct sum

$$\begin{aligned} U_{t-1,i} &= U_{t-1,i-1} \oplus \mathcal{A}'_{v(i,1)}(t-1) \oplus \dots \oplus \mathcal{A}'_{v(i,r'_i)}(t-1) \\ &= U_{t-1,i-1} \oplus \mathcal{A}_{v(i,1)}(t-1) \oplus \dots \oplus \mathcal{A}_{v(i,r'_i)}(t-1), \end{aligned} \quad (22)$$

- $p_{v(i,1)} = \dots = p_{v(i,r'_i)} =: b_i$ ,
- $b_i \neq b_j$  if  $i \neq j$ .

We suppose that the summands in (19) are numbered such that

$$\mathcal{A}_{v(i,1)}(t) \neq 0, \dots, \mathcal{A}_{v(i,k_i)}(t) \neq 0, \quad \mathcal{A}_{v(i,k_i+1)}(t) = \dots = \mathcal{A}_{v(i,r'_i)}(t) = 0. \quad (23)$$

Let us prove (i)–(iv). Consider two cases that differ in the direction of the last arrow in (9).

Case 1:  $\alpha_{t-1} : (t-1) \longrightarrow t$ . By (12),

$$U_{t1} = A_{t-1}U_{t-1,1}, \dots, U_{t,t-1} = A_{t-1}U_{t-1,t-1}, \quad U_{tt} = U_t.$$

By (22), (23), and (21), we have

$$U_{ti} = \begin{cases} U_{t,i-1} \oplus \mathcal{A}_{v(i,1)}(t) \oplus \dots \oplus \mathcal{A}_{v(i,k_i)}(t) & \text{if } i < t, \\ U_{t,t-1} \oplus \mathcal{A}_{s'+1}(t) \oplus \dots \oplus \mathcal{A}_s(t) & \text{if } i = t, \end{cases}$$

which is the desired decomposition (20).

Case 2:  $\alpha_{t-1} : (t-1) \longleftarrow t$ . By (12),

$$U_{t1} = \text{Ker } A_{t-1}, \quad U_{t2} = A_{t-1}^{-1}U_{t-1,1}, \dots, U_{tt} = A_{t-1}^{-1}U_{t-1,t-1} = U_t.$$

By (21), (22), and (23), we have

$$U_{ti} = \begin{cases} \mathcal{A}_{s'+1}(t) \oplus \dots \oplus \mathcal{A}_s(t) & \text{if } i = 1, \\ U_{t,i-1} \oplus \mathcal{A}_{v(i-1,1)}(t) \oplus \dots \oplus \mathcal{A}_{v(i-1,k_{i-1})}(t) & \text{if } i > 1, \end{cases}$$

which is the desired decomposition (20).

In both the cases, the family of pairs  $\{(p_v, q_v)\}_{v=1}^s$  (which is determined up to renumbering) can be obtained from  $\{(p_v, q'_v)\}_{v=1}^{s'}$  by replacing  $k_i$  pairs  $(a_i, t-1)$  with  $(a_i, t)$  for each  $i = 1, \dots, t-1$  and by attaching  $k_t := s - s'$  pairs  $(t, t)$ . This proves the statement (i) since  $k_1, \dots, k_t$  are expressed via  $n_{ij}$ :

$$k_i = \dim U_{ti} - \dim U_{t,i-1} = n_{ti} - n_{t,i-1}, \quad i = 1, \dots, t$$

(we set  $n_{t0} := 0$ ).

The statements (ii)–(iv) follow from the induction hypothesis and Cases 1 and 2.  $\square$

#### 4. Topological isomorphism of chains

The goal of this section is to prove Theorem 1.

Let  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$  be a topological isomorphism of chains of the form (1) and (2). Due to Theorem 4, it suffices to prove that their sets (18) coincide; that is,

$$\dim U_{ij} = \dim V_{ij} \quad \text{for all } i, j,$$

in which  $U_{ij}$  are the vector subspaces of  $U_i$  that were constructed in (12), and  $V_{ij}$  are the vector subspaces of  $V_i$  that are analogously constructed by the chain  $\mathcal{B}$ . Due to Definition 1(iii), the topological isomorphism  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$  is formed by the homeomorphisms  $\varphi_i : U_i \rightarrow V_i$ . It suffices to show that



each  $\varphi_i$  maps  $U_{ij}$  on  $V_{ij}$  since then each  $U_{ij}$  is homeomorphic to  $V_{ij}$  and by [12] all homeomorphic vector spaces have the same dimension. What is left is to prove the following lemma.

**Lemma.** If  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$  is a topological isomorphism of chains (1) and (2), then

$$\varphi_i U_{ij} = V_{ij} \quad \text{for all } i = 1, \dots, t \text{ and } j = 1, \dots, i. \quad (24)$$

**Proof.** The assertion (24) holds for  $i = 1$  since  $\varphi_1 : U_1 \rightarrow V_1$  is a bijection. Suppose that (24) holds for  $i = k$  (and all  $j = 1, \dots, k$ ); let us prove it for  $i = k + 1$ . It suffices to prove that

$$\varphi_{k+1} U_{k+1,j} \subset V_{k+1,j} \quad \text{for all } j = 1, \dots, k + 1 \quad (25)$$

since then we can use (25) for  $\varphi^{-1} : \mathcal{B} \xrightarrow{\sim} \mathcal{A}$  instead of  $\varphi$  and obtain

$$\varphi_{k+1}^{-1} V_{k+1,j} \subset U_{k+1,j} \quad \text{for all } j = 1, \dots, k + 1,$$

which ensures  $\varphi_{k+1} U_{k+1,j} \supset V_{k+1,j}$ .

In the case  $A_k : U_k \rightarrow U_{k+1}$ , the inclusion (25) holds since if  $y \in U_{k+1,j}$  and  $x \in A_k^{-1}y \subset U_{kj}$ , then

$$\begin{array}{ccc} U_{kj} & \xrightarrow[A_k]{\text{on}} & U_{k+1,j} \\ \varphi_k \downarrow \text{biect} & & \downarrow \varphi_{k+1} \\ V_{kj} & \xrightarrow{B_k} & V_{k+1} \end{array} \quad \begin{array}{ccc} x & \xrightarrow{\quad} & y \\ \downarrow & & \downarrow \\ \varphi_k x & \xrightarrow{\quad} & B_k \varphi_k x \end{array}$$

Thus,  $\varphi_{k+1}y = B_k \varphi_k x \in B_k V_{kj} = V_{k+1,j}$ .

In the case  $A_k : U_k \leftarrow U_{k+1}$ , the inclusion (25) holds since if  $y \in U_{k+1,j}$  then

$$\begin{array}{ccc} U_{kj} & \xleftarrow{A_k} & U_{k+1,j} \\ \varphi_k \downarrow & & \downarrow \varphi_{k+1} \\ V_k & \xleftarrow{B_k} & V_{k+1} \end{array} \quad \begin{array}{ccc} A_k y & \xleftarrow{\quad} & y \\ \downarrow & & \downarrow \\ \varphi_k A_k y & \xleftarrow{\quad} & \varphi_{k+1} y \end{array}$$

Thus,  $\varphi_k A_k y \in V_{kj}$  and so  $\varphi_{k+1}y \in V_{k+1,j}$ .  $\square$

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